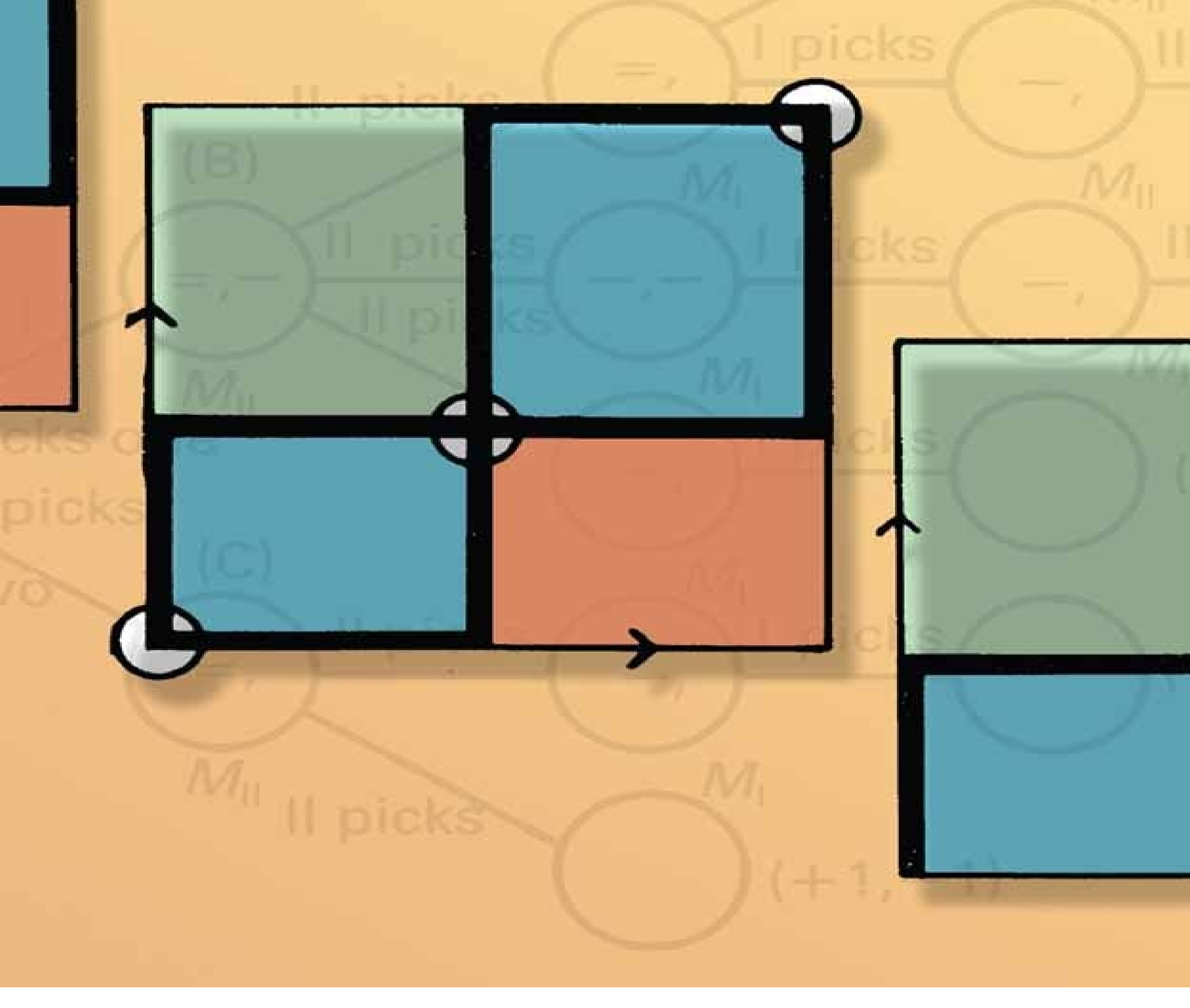
GAMES, THEORY AND APPLICATIONS

L. C. THOMAS



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To Carey, Iris, Fred and Glenys,

and in memory of Eunice

Author’s Preface

The preface of a book is positioned first, written last, and read little. It should also be the author guide to and *raison d’être* for the following manuscript. So here goes.

The theory of games, as a way of modelling problems with two or more decision-makers involve originated at the end of the Second World War, though the playing of games themselves has been on of the enduring strands in the history of mankind. At the same time as game theory was bein developed, new mathematical techniques were being invented in other types of decision-making. Th whole area of ‘applicable mathematics’ has continued to be one of the most active branches research and development up to the present day. Game theory, despite an uneven reputation durin this period, has continued to develop, in depth of understanding and breadth of applications.

Its fascination, for me, is that it is the one area where quite difficult logical ideas can be express rigorously, with the minimum of mathematical sophistication. These ideas can be quick comprehended because there are manifestations of them in real life, with which people are familia Thus, of itself, game theory has much to recommend it, apart from its applications. However, it is th applications which are of paramount importance. Games can be used as models of situations diverse as international conflict, advertising budgets, pricing policy, airport loading fees, and th evolution of animal behaviour. In some cases, the analysis of the game will lead to a specific strateg for each participant being considered as his best way of playing the game: in other games and types analysis, all that can be done is an exploration of what were the players’ understanding of each oth in order for a certain outcome to occur. So there is something for everyone.

The book evolved from a lecture course I have given for several years to undergraduates Manchester. The students came from many different departments—mathematics, science, economic accountancy, liberal studies, and psychology. Thus, the book assumes a modest mathematic background—an ability to differentiate and integrate simple functions, the idea of maxima an minima of functions, familiarity with matrix notation (though not the properties of matrices) an summation signs, and a basic knowledge of probability. Although my upbringing means I like to u theorems and lemmas to structure and point out the important results, proofs are only given whe they can be understood with this background, and employ ideas and techniques which are used aga in the text.

The book’s aim is to give an introduction to game theory to as wide a numerate audience possible. Thus, I do not concentrate on linear programming solution techniques, which would weight more towards the mathematician; and I treat the economic applications of game theory, which sever authors have concentrated on, as only one of several such applications.

Chapter 1 introduces the terminology of game theory and some examples of games which will rec throughout the book. Chapters 2, 3, and 4 give the theory of two-person and *n*-person games. Th remaining chapters are independent of one another, but do use the theory developed in the first fe chapters. However, I do point out connections between work in different chapters when they arise. A chapters end with a section of further reading which indicates where one can pursue the ideas of th chapter further, and some problems. I have included solutions to the problems to encourage the read to attempt them. Doing the problems is the way of proving you understand the ideas in the text.

I am grateful to my colleagues Doug White, Roger Hartley, and Chris Birchenall for many usef conversations. I am especially grateful to Simon French, who taught a course from the draft of th book for a year, and whose comments and corrections of my errors proved very valuable. A remaining errors are ‘deliberate mistakes’ by me! My thanks are also due to Kate Baker and Bo Lande, who typed the manuscripts with great care, on both sides of the Atlantic in between babies ar racquetball championships, respectively—anything to get away from my handwriting! Finally, m thanks go to my family who taught me the best cooperative game of all while I was writing this book

L. C. Thom

Glossary of Symbols

$x \in X$	means x is a member of the set X
$x \notin X$	means x is not a member of the set X
$\{x\vert ....\}$	is the set of x with the property ...
$S \cup T =$	$\{x\vert x \in S$ or $x \in T\}$
$S \cap T =$	$\{x\vert x \in S$ and $x \in T\}$
$S \subseteq T =$	set S is included in set T
$S - T =$	$\{x\vert x \in S$ and $x \notin T\}$
$\emptyset$	is the empty set
$\#S$	the number of elements in the set S
$\vert x\vert =$	$\max \{x, -x\}$
$x > y$	x greater than y
$\mathbf{x} \geqslant \mathbf{y}$	x greater than or equal to y
$x \not> y$	x not greater than y
$n\,!$	1. 2. 3. 4. .... $(n-1)$. n

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \dots + x_n$$

If $\underline{\mathrm{x}} = (x_1, \dots, x_n)$

$$\|x\| = \max_{k} |x_k|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = f'$ is the derivative of function f

Two-person zero-sum games

I, II are the players

I_i (II_j) ith pure strategy of player I (II)

e_{ij} payoff to player I of I_i versus II_j

v_{L}, v_{U} lower and upper values of the game

$\mathbf{x} = (x_1, ..., x_n)$ mixed strategy for player I

$\mathbf{y} = (y_1, y_2, ..., y_m)$ mixed strategy for player II

v value of game

$\mathbf{x}^*(\mathbf{y}^*)$ optimal strategy for player I (II)

$X(Y)$ set of all mixed strategies for player I (II)

Two-person non-zero-sum game

$e_i(\mathbf{x},\mathbf{y})$	payoff to player i if I plays mixed strategy x and II plays mixed strategy y
v (v_{II})	maximin values for player I (II)

B –	bargaining set

n-person games

$N = \{1, 2, ...n)$	is the set of players
$S \subseteq N$	coalition of players
$v(\cdot)$	characteristic function
X_S	set of coordinated strategies for players in S
$x = (x_1, x_2, ..., x_n)$	imputation
$E(v)$	set of imputations in game with characteristic function v
$C(v)$	core of game with characteristic function of v
$N(v)$	nucleolus of game with characteristic function of v
$S(v)$	stable set of game with characteristic function v
$\phi_i(v)$	Shapley value of player i in game with characteristic function v
$x \underset{S}{\geqslant} y$	imputation x dominates y on coalition S
$x > y$	imputation x dominates imputation y

metagames

$k_1k_2 \ldots k_nG$	the metagame $k_1k_2 \ldots k_nG$ based on game G
$R_i(k_1k_2 \ldots k_nG$	rational outcomes for player i in $k_1k_2 \ldots k_nG$
$\hat{R}_i(k_1k_2 \ldots k_nG)$	metarational outcomes for player i in $k_1k_2 \ldots k_nG$
$\hat{E}(G)$	metaequilibria in game G

multi-stage games

Γ_i	ith subgame of multi-stage game Γ
$v = (v_1 \ldots v_n)$	values of game Γ
val $\Gamma(w)$	value of game $\Gamma(w)$
h_k	history of kth stage of the game
H_k	history of game up to kth stage
$E_i(\cdot,\cdot)$	payoff to player i in supergame

evolutionary games

$e(x, y)$	fitness payoff of playing strategy x against a population playing strategy y
$e_i(r,(x, y))$	fitness payoff to player of type i of playing strategy r against a population where type 1's play x and type 2's play y

CHAPTER 1

‘The game’s afoot’

(Shakespea

1.1 WHAT IS A GAME?

Have you read the newspaper today? If so, the front page probably contains a report of some politic controversy, or a strike, or perhaps describes armed conflict and violent actions by groups of people c countries. The inside pages will report actions by various pressure groups to change social policy, c will describe government decisions about such policies—improvements in housing, allocation finance to the various branches of health and social services. The financial pages will be full of tak overs, firms’ attempts to improve their market share, changes in prices of goods, or governme attempts to control the financial markets. Finally, there will be the sports page or the chess or bridg column to read.

What do all these reports have in common? They all describe conflicts of interests between peop or groups of people such as political parties, governments, and businesses. We call the theoretic models of such conflicts of interests **games.** Recalling the famous definition of a model as a sma imitation of the real thing (as in ‘model husband’) indicates that in a game one is trying to extract th essential problems in the conflict of interest. **Game theory** consists of ways of analysing the problems. Obviously, most conflicts are too complicated to include all the facets involved in th corresponding model, but a game could still be useful in describing the main types of decisions ope to the participants and the sort of results that could occur. For some games, game theory will suggest ‘solution’ to the game, that is a best way of playing the game for each person involved; but for mo games describing real problems all it can do is to rule out some types of decisions and perhap suggest, which players will work together.

These theoretical models of conflict are called games because we can identify easily the conflicts interest in recreational games like Poker, Noughts and Crosses, or Monopoly; and some of the boar games, like Chess, did develop historically as models of warfare. It is in a way an unfortunate choic of name, because it has the connotations of amusement, light-heartedness, and a recreational contest hope the reader will occasionally be amused, but games cover a much wider area than just boa games or ‘bored’ games. We shall consider economic and business problems, the tactics and logistic of warfare, international and national politics, and social policy all as candidates to be modelled games.

1.2 EXAMPLES OF GAMES

To get a feeling for what is involved in a game, let’s look at some simple examples. Try to think wha are their important features and what if anything they have in common.

Example 1.1 — Noughts and Crosses. Draw an $n \times n$ grid. The first player puts a nought in ar square, and then the second player puts a cross in some unused square. Continue in sequence un one player has a line (column, row, or diagonal) of n noughts or n crosses. That player then wins.

You probably haven't played this game for a few years, so now is your chance. You will find that i n = 1 or 2, the first player always wins, and if n = 3, you can always draw unless you make a mistak What happens for n = 4 and higher n? What about the three-dimensional version, i.e. $n \times n \times n$?

Example 1.2—Simplified Poker. There are only two cards involved—an 'Ace' and a 'Two'—an only two players—whom we label I and II. Each puts £1 in the pot and I deals II one card, which then looks at. If it is the Ace, II must say 'Ace', but if it is the 'Two', II can say 'Ace' or 'Two'. I he says 'Two' he loses the game and his £1 in the pot. If, however, II says 'Ace' no matter what th card is, II must put another £1 in the 'pot'. In this case player I can either believe him, i.e. 'fol and so lose his £1 in the 'pot', or else he can demand to 'see' the card. In this case I has to p another £1 into the pot, and then the card is shown. If II had the Ace he wins the 'pot' and so take £2 from I, but if II has the Two, I wins the 'pot' and so takes £2 from II. This game involves th elements of 'bluffing' and 'calling' involved in real poker, but not surprisingly has not yet take Las Vegas by storm.

Example 1.3 — Nim. A number of matches are set out in two piles, and two players take turns taking matches from the piles. At each turn a player must take at least one match, but he can tal more provided they are all from the same pile. The loser is the player who picks up the last match.

Example 1.4—Prisoners' Dilemma. Two people, arrested with stolen property in their possessio are being interviewed separately by the police. They both know that if they keep quiet there is n enough evidence for them to be convicted of theft, and so they will only get a one-year ga sentence for being in possession of stolen property. If both confess to the theft, they will both g nine years in prison. However, if one confesses and the other keeps quiet, the one who turne Queen's Evidence will go free, while the other will have a ten-year gaol sentence (the extra year for not assisting the police). What should they do? Would it make any difference if they could tal to each other between being arrested and interviewed? You can argue that this game also embodie the dilemma over nuclear disarmament or whether unions should pursue high or low wage clain for their members. Can you see why?

Example 1.5—Pick a number. Each person in a group of people chooses a number. The one with th highest number wins. What number would you choose? Infinity is not allowed. Could you guarante to win the game somehow?

Example 1.6—Duellists. Two duellists stand $2N$ paces apart with loaded pistols and start to wa toward each other. At each pace they can decide whether or not to fire their one bullet, and th chance of killing their opponent increases as they get nearer. If they fire and miss, honour demand that they still keep walking nearer. When should each man fire? Is this affected by whether the aim [*sic*] is to kill their opponent or to stay alive themselves? In a duel, you would know when yo opponent had fired and missed, but suppose instead you were in planes, far apart, armed with missile, so you didn't know if your opponent had fired and missed. Would this make any differenc to when you should fire?

1.3 TERMINOLOGY OF GAME THEORY

Let us emphasise again that game theory is not a prescriptive way of how to play a game. Rather it is set of ideas and techniques for analysing these mathematical models of conflict of interest. It doesn tell you how to play the game, but describes properties that certain ways of playing the game hav and which you might think desirable. Even when the analysis suggests a best way of playing the gam it only does it assuming that everyone is playing in the 'best way' they can. It never allows for ways punishing your opponent if he makes a mistake, which is the way most games, whether board ones real life conflict situations, end.

Before we start on this analysis let us look again at the six examples in the previous section, and s what features they have in common. This will help us in defining the terminology of game theory.

Firstly there are always at least two participants (as many as you like in Example 1.5) called t **players** hereafter labelled I, II, III, etc. or, in later chapters, 1, 2, 3, etc. Each game consists of sequence of **moves,** some simultaneous, which are either decisions by the players or outcomes chance events. Thus, in the simplified poker of Example 1.2 the first move is the chance event which card is picked. If it is a Two, this is followed by the decision by II whether to say 'Ace' 'Two'. If II says 'Ace', the last move is whether I believes him or not.

At the end of the game, each player receives a **payoff.** We will always assume that the payoff given by a real number. In many games you could associate this number with the amount won, or s the payoff is +1 if you won the game, 0 if it is a draw, and –1 if you lose it. However, in other gam the result is more complicated or intangible.

When von Neumann and Morgenstern (1947) introduced the basics of game theory, they al developed the idea of the **utility** of an outcome of the game, so that these numbers reflect yo preferences. Thus, suppose the outcomes of the game were 'going to a football match' or 'going to t cinema' and you preferred the football match. You would choose two numbers u(FM)—the utility going to the football match—and u(C)—the utility of going to the cinema—so that $u(\text{FM}) > u(\text{C})$. S $u(\text{FM}) = 4$, $u(\text{C}) = 2$. Obviously you can choose almost any pair of numbers here, but if you sta asking more of the utility function this cuts down the number of possible numerical representatio Thus, you may prefer seeing the football match in the dry, to going to the cinema; but would rather to the cinema than get soaked watching a football match. This requires the utilities to satisfy

$$u(\text{FM Dry}) > u(\text{C}) > u(\text{FM Wet}). \qquad (1.$$

If you think the chance of rain tonight is ½, and you can't decide whether to go to the football mat or the cinema (you are indifferent) this would be represented by the equation

$$\tfrac{1}{2}u(\text{FM Dry}) + \tfrac{1}{2}u(\text{FM Wet}) = u(\text{C}). \qquad (1.$$

Thus, if you choose $u(\text{FM Dry}) = 4$, $u(\text{C}) = 2$, this means u(FM Wet) must equal 0. This is the situati in most versions of utility theory, where you can choose the utility of two outcomes as you lik provided they reflect your preference, but the utilities of all the other outcomes are then fix completely by your preferences. Readers who want to learn more about utility theory and the axio underlying it should turn to Chapter 2 of Luce and Raiffa (1957) or to Raiffa (1968).

We will always assume that the players' preferences over the outcomes of a game satisfy the rul underlying utility theory, and so each outcome can be represented numerically by its utility valu Remember that this utility reflects all the aspects of the outcome, including your regret or joy abo what happened to your opponent. So if you get a payoff of utility value 4 and your opponent gets o of 2, you are as happy with that as if you get 4 and he gets 200 or –200.

Returning to the common features in the various games, notice that each player has to ma decisions at some moves of the game. A **strategy** for a player is a description of the decisions he w make at all the possible situations that can arise in the game. Thus, having chosen his strategy it w tell him what to do at every situation that can arise no matter what his opponent does or what are t outcomes of the chance events. In 2 × 2 Noughts and Crosses, a strategy tells the first player which the ~~four squares to put his first nought in; and for each of the three replies by his opponent it mu~~st sa which of the remaining two squares to put the second nought in. Think of it as a set of instructio which enables a computer to play the game for you. It is obvious that for many games, chess f instance, although in principle you can conceive of a strategy, in practice it is too long to write dow (If you were 'Black' in chess you would have to write down your response to all 20 possible openi moves by White, and at your second move reply to all the 400 different situations White can be after two moves, and so on.)

If the sum of the players' payoffs is zero no matter what strategy they use, the game is called **zer sum**. In these games, like Nim, or Poker (Examples 1.2 and 1.3), the players are completely oppos to one another in that what one wins the other loses. Games which don't have this property are, n surprisingly, called **non-zero-sum** games. In Prisoners' Dilemma the total payoff is two years prison if they both keep silent and 18 years if they confess.

Finally, if at every move in the game all the players know all the moves that have already occurre the game is said to be one of **perfect information.** Thus, Noughts and Crosses and Nim have perfe information, whereas in Poker player I doesn't know which card player II has picked up. Later w shall show that this difference leads to a difference in the type of strategy you might think is best each game. Is it a good idea always to play the same strategy in Poker? What about Noughts a Crosses?

1.4 HISTORY OF GAME THEORY

Game theory started with two papers by von Neumann (Von Neumann, 1928, 1937), though Borel i the 1920s had also looked at similar problems. (See Borel, 1953, for a translation.) It really sprang life, however, with the publication of von Neumann and Morgenstern's book *Theory of Games a Economic Behaviour* in 1944, and especially the second edition in 1947. Rarely has the first book in subject contained so many of the ideas that are still the main areas of interest in the subject or made great an impact as this one. The reason was mainly the Second World War, because during this ther had been considerable activity in modelling decision situations which involved one or more decisio makers. Hence, the rise of Operations Research. Most of the military problems that can be modell as games are of the two-player zero-sum type, and these are the very ones for which game theory c suggest a specific 'solution'. Thus, at the end of the war, people were thinking of how to mod decision situations, and there was a view that game theory had had a successful, if secret, track reco in the military area.

In the next few years there was a great deal of work in game theory as people sought to show it w the mathematical panacea in all areas of human conflict. They tried to expand the mathematics incorporate other problems, such as bargaining and arbitration, into the framework. They tried to sol the open theoretical question, and to apply the theory to areas like politics and economic competitio The four volumes: *Contributions to the Theory of Games* (Kuhn and Tucker, 1950, 1953; Dresche Tucker and Wolfe, 1957; and Tucker and Luce, 1959), give a good idea of the problems that we being examined at that time.

However, as people concentrated more on games with more than two people, it was realised th there was no 'nice' solution concept for these games; nor did the game models capture all the featur of real-life conflict situations. Thus, when in 1957 Luce and Raiffa wrote the other classic book in t area, *Games and Decisions*, they were careful to point out the limitations of game theory. This fa from grace continued through the 1960s, and the nadir was reached when Lucas (1967) found a te person game that did not have a 'solution' in the sense suggested by von Neuman and Morgenstern.

Since then game theory has recovered some of its popularity. It is no longer considered th ‘panacea’ to solve all human conflict, but it is certainly the best way of thinking about confli situations. It is a useful test-bench for looking at what new concepts about conflict actually imply, an it highlights the important decisions that have to be made in real-life situations. There are still larg groups of researchers, especially in the USA and Israel, working on ‘classical’ game theory an bargaining. This involves developing and exploring new properties that they feel resolutions conflict situations should possess, and widening the areas of application of game theory even include religion (Brams, 1980).

There has also been tremendous development in using game theory to record how people actual react in conflict situations. Psychologists and game theorists devise particular games to test wh people’s decisions are in these situations and how they compare with game theory prediction Prisoners’ Dilemma is a popular candidate for such a game, and the number of published pape concerning experiments with it is over two hundred. This has led to the idea of devising games as teaching and learning tool, and in Chapter 10 we shall look at **gaming,** as this aspect of game theory called. Another development from the experiments of actually playing games is to try to explain ho people arrive at outcomes which are not in agreement with the ‘good’ solutions of classical gam theory. The most obvious of these theoretical extensions is the idea of metagames introduced b Howard (1971).

In the last ten years completely new applications of game theory have been developed unexpected areas. Thus, Maynard-Smith (1974) described how a game-theoretic framework is usef in describing the evolution of genes, which affect breeding patterns.

Thus, games and game theory can look forward to an exciting future, not as a way of solving a conflict problems, but as the most useful collection of techniques for analysing these problems.

PROBLEMS FOR CHAPTER 1

1.1. Consider the following types of games.

(a) Games with perfect information, where some of the moves are chance events.
(b) Games with perfect information, with no chance moves.
(c) Games which do not have perfect information, but have chance moves.
(d) Games which do not have perfect information and do not have chance moves.

Which type is each of the following games Chess, Bridge, Monopoly, Draughts (Checkers), Scissors Stone–Paper, Ludo, Poker, Noughts and Crosses? Write down one more game of each type.

1.2 Give examples of a situation (not necessarily a board game) which can be modelled as a game

(a) with two players and is zero-sum;
(b) with two players and is non-zero-sum;
(c) with more than two players and is zero-sum;
(d) with more than two players and is non-zero-sum.

1.3. Explain why there are really two strategies for each player in 2 × 2 Noughts and Cross (Example 1.1). What are they?

1.4. In the duellists’ game (Example 1.6), write down all the strategies for each player in a ‘quie duel when the players don’t know if their opponent has fired. What is the difference betwe these strategies and the ones in a ‘noisy’ duel where they know if their opponent has fired?

1.5. Let $u(x)$ be a person’s utility function of wining £x: and it is standardised by calling $u(0) = 0$ an $u(100) = 100$. If he is indifferent between winning £40 for certain or taking part in a gamb where he has a 50% chance of winning £100 and a 50% of winning £0, what is $u(40)$? If he willing to pay £10 to take part in a gamble where you have a ¼ chance of winning £50, otherwi you get no prize, what can you say about his utility of losing £10?

TWO-PERSON ZERO-SUM GAMES

'The game is never lost till won' *George Cra*

2.1 EXTENSIVE FORM

We concentrate in this chapter on games with only two players, I and II, where what I wins, II lose i.e. a two-person zero-sum game. One way of describing such a game (in fact it will work for ar game) is by recording all possible sequences of moves that can occur in it, and the payoff at the end each sequence. This can be represented diagrammatically by a **tree graph**. In such a graph each poi represents a point in the game where a move must be made. Remember, a move is either a decision b one of the players or a chance event. The possible moves that can be made at that point (either th outcomes of the chance event or the possible decisions by the player) are represented by lines drav from that point (Fig.2.1).

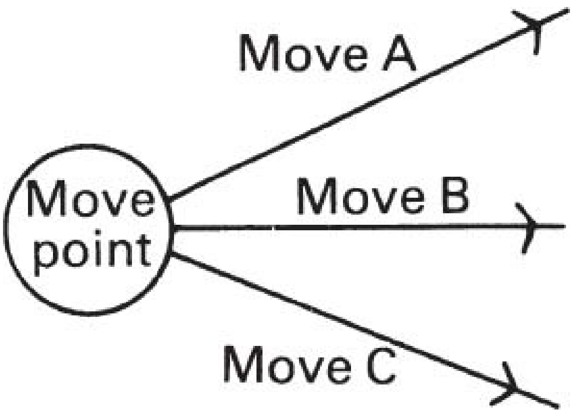

Fig. 2.1—Typical move on a tree graph.

If we can arrive at the same point in the game by two different sequences of moves, we wi represent it by two different points, since this allows the decisions made to depend on what h already happened in the game, and so be different for the two different sequences. Thus, we cann have the situation shown in Fig. 2.2.

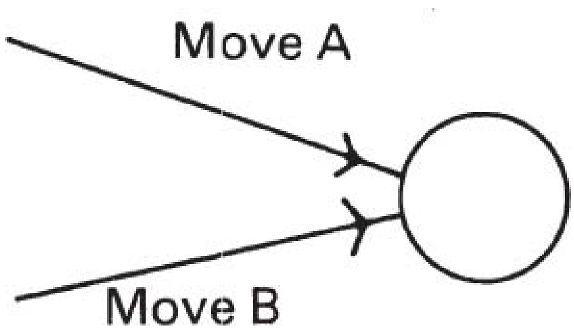

Fig. 2.2—This cannot happen.

Let's look at Examples 1.2 and 1.3 in this way, which is called the **extensive** form of the game.

Example 2.1—Simplified Poker (as in Example 1.2). In the tree graph for this game we label t chance move points M_0; those that involve a decision by player I, M_I; and those that involve decision by II, M_{II} (see Fig. 2.3). The payoffs are written (a, b) where a is I's payoff and b is II' Looking at the tree, one can see that I has two strategies, namely:

I_1—believe II when he says Ace,

I_2—don't believe II when he says Ace;

while II also has two strategies:

II_1 — say 'Two' when he has a 'Two',

II_2 — say 'Ace' when he has a 'Two'.

The real crux of the game is that player I doesn't know which of the M_I points he is at when he h to make his decisions. Given the information I has available he cannot distinguish between the points. Points with this property are said to be in the same **information set**, and all a player know is that he is in that set somewhere.

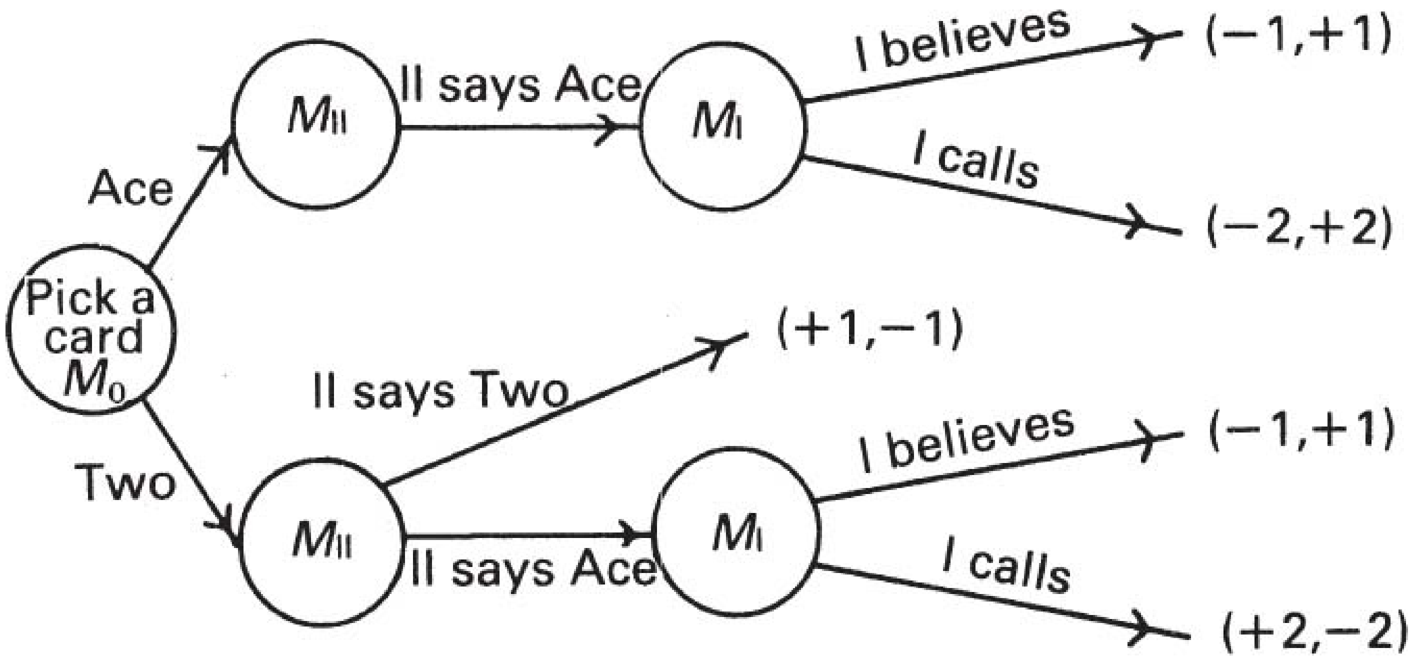

Fig. 2.3—Extensive form of Poker.

Example 2.2—2-2 Nim (as in Example 1.3). Take the very simple example where there are tw matches in each pile. This example was discussed in Jones (1980) and leads to the tree graph show in Fig. 2.4, where (=, =) is the situation with two matches in each pile. From the tree, we can aga identify the different strategies for each player—three for player I, and six for player II— follows:

I_1 — take 1 match in the (=, =) case and 1 in the (=,) case,

I_2 — take 1 match in the (=, =) case and 2 in the (=,) case,

I_3 — take 2 matches in the (=, =) case.

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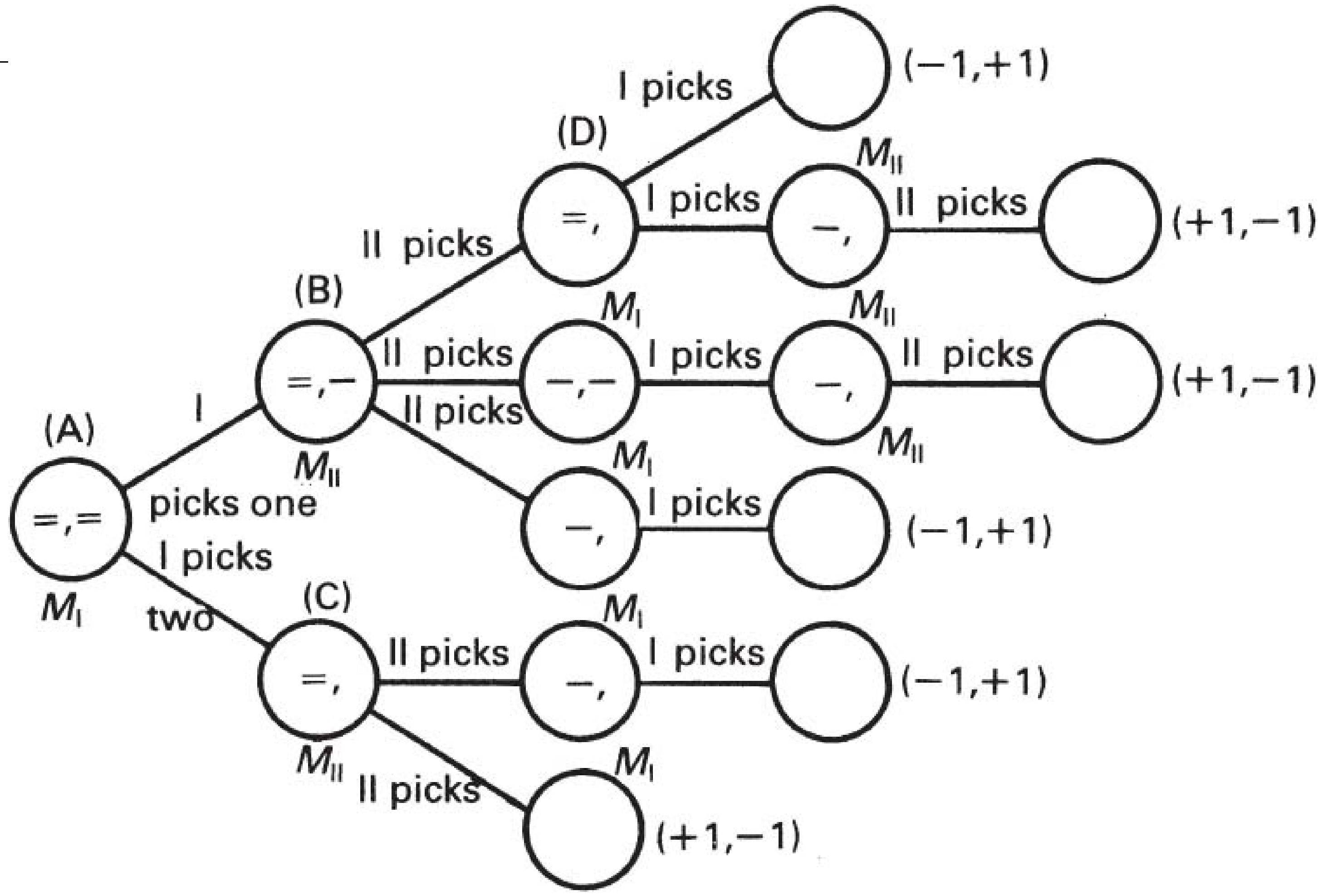

Fig. 2.4.—

Notice that in I_3 you do not have to state what to do in the (=,) case since it will not arise for I. II
strategies are:

II_1 — if (=,) take 2 matches, and if (=, −) take 1 from smaller pile,
II_2 — if (=,) take 2 matches, and if (=, −) take 1 from larger pile,
II_3 — if (=,) take 2 matches, and if (=, −) take 2 from larger pile,
II_4 — if (=,) take 1 match, and if (=, −) take 1 from smaller pile,
II_5 — if (=,) take 1 match, and if (=, −) take 1 from larger pile,
II_6 — if (=,) take 1 match, and if (=, −) take 2 from larger pile.

Also in this problem we can prune the tree somewhat by working back from the end of each branc and working out what the best decision at each move is. Thus, at point D, one of I's strategies leads t a payoff– 1 and the other to a payoff + 1 for him, so he would obviously choose the latter. Thus, w can remove the branches of the tree from D onwards and replace them by the payoffs (1,−1). So at I II's decisions now lead to payoffs –1, –1 and +1, respectively, for him, so he would obviously choos the last. Thus, we can replace the tree from B onwards by the payoffs (−1, +1). Similarly at C, would choose the decision that leads to (−1, +1). The tree now looks like the one shown in Fig. 2.5. is obvious no matter what I does his payoff will be −1, i.e. II will win the game.

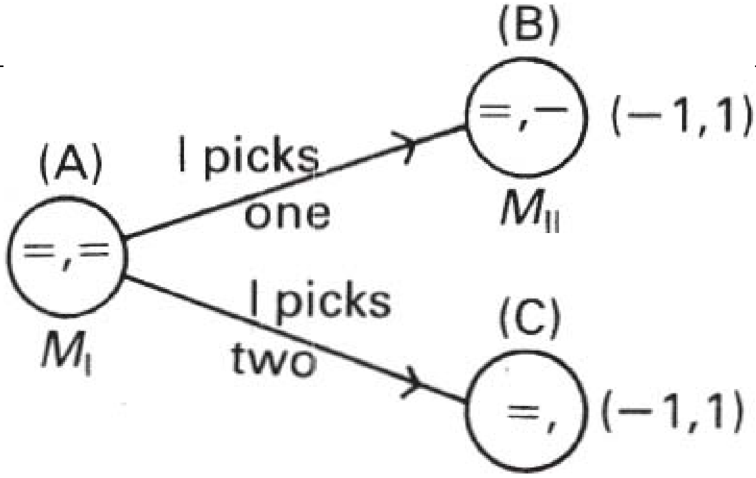

Fig. 2.5—Pruned tree for Nim.

Thus, in Nim we can use the extensive form to work out who will win the game. Notice that all th information sets consists only of one point. In Poker, we can't say who would win because we can work back along each branch separately. What I chooses between 'believes' and 'calls' must hold fo both branches. This difficulty arises because Poker is not a game of perfect information like Nim an so has information sets consisting of more than one point.

These examples highlight the advantages and disadvantages of the extensive form of games. Th disadvantages are firstly, even for Nim, it was quite complicated to draw the tree and then work bac through it. For most games it is just too long to draw the tree graph and work back through it to fin the answer to the game. Secondly, for games like Poker, which don't have perfect information, yo cannot work back through the tree because you have to make the same decision on different branch of the tree. The advantage of the extensive form is that it enables you to get a 'picture' of the game s that you can systematically work out what are the possible strategies. Also, it seems obvious that fo games of perfect information we could in principle always draw the tree graph and then work bac through the tree to find the solution to the game, i.e. a best strategy for each player and the expecte payoff if these play each other. Chess, Draughts and Noughts and Crosses all have perfect informatio What is their solution?

2.2 NORMAL FQRM

Another way of analysing games starts by listing all possible strategies for each player, i.e. I_1, I_2, . . . I_n for player I and II_1 II_2, . . ., II_m for player II. We could use the tree graph to find all these strategie Although, as we mentioned above, this could be a very difficult if not impossible task, we can alway make this list in principle, and so this practical problem will not invalidate any of the theoretic results obtained. Having made the list, if we imagine I playing strategy I_i, and II playing II_j, we ca work through which moves each player will play at each decision point and get to the payoffs, whic we write as $(e_{ij}, -e_{ij})$. Thus, in Nim (Example 2.2) if I plays I_1 and II plays II_1 firstly I takes 1 matc leaving (=, –) then II takes 1 from the smaller pile to get (=,). I_1 then tells I to take one more matc leaving (−, .) and II must pick up the last match and so lose the game. So the payoff is (1,−1).

If there are chance moves in the game, then when I plays I_i against II's II_j, the payoff depends o the outcome of the chance move. It seems reasonable in such a case to take the expected payoff whe I_i plays II_j. So we multiply each payoff by the probabilty of the chance event that gave rise to it, an add all these products together. This gives the average or expected payoff. In Poker (Example 2.

suppose I_1 plays II_2. Then if the outcome of the chance event is that II gets the Ace, he says 'Ace' an I believes him. The payoff then is (−1,+1). If the outcome of the chance event is that II gets the Tw however, he says 'Two' and loses the game giving payoff (+1, −1). We assume the chances of the Ac and Two being chosen are both ½, so the expected payoff is ½(−1,+1) + ½(+l,−1) = (0,0). Notice th once we start using expected payoffs, we are implicitly thinking of the game being played over an over again, and looking at the long-run average. We will return to this point below.

Thus, we can think of the game as one in which each player simultaneously (or in secret from th other) chooses one of his strategies, I_i and II_j say, and the expected payoff is then $(e_{ij}, -e_{ij})$. This is th **normal form** of the game. From now on because Payoff II = −Payoff I in this chapter, we shall on record I's payoffs. We record them as an $n \times m$ matrix with the i, jth entry, e_{ij}, the **payoff matrix**. the normal form of the game you can make all the moves you could make in the original game, an get the same payoff although the sequential nature of the moves has been lost, as has the idea perfect information. However, it is the same game we are looking at, whichever form we tak Returning to the two examples of Section 2.1 again and writing them in normal form, we get th following:

Example 2.1 — Simplified Poker. Recall:

I_1 — believe II when he says 'Ace',

I_2 — don't believe II when he says 'Ace',

are I's strategies, and II's are:

II_1 — say 'Two' when you have a Two,

II_2 — say 'Ace' when you have a Two (bluff).

The payoff matrix is:

$$\begin{array}{c c} & \begin{array}{cc} II_1 & II_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{pmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{pmatrix}. \end{array} \qquad (2.$$

This is obtained as follows. For I_2 versus II_1 if II gets an Ace he says 'Ace' and I believes him, so has payoff − 1; if II gets the Two he says 'Two' and so I gets payoff + 1 straightaway. The chance o an Ace is ½, the chance of a Two, is ½ so expected payoff is 1.½ + −1.½ = 0. For I_1 versus II_2, n matter what II gets, he always says 'Ace' and I always believes him and gets payoff − 1. In I_2 again II_1, if II has an Ace he says 'Ace', but I does not believe him and so when Ace is shown I loses − whereas if II gets the Two he says so and I gains +1. The expected payoff is ½ × −2 + ½ × 1 = − Lastly if I_2 plays II_2, then II always says 'Ace', and I never believes him. When the card is shown, if is an Ace, I gets −2, but if it is a Two he gets +2. Thus, the expected payoff is −2 × ½ + 2 × ½ = 0.

Example 2.2 **Nim**. Recall that I has three strategies.

I_1 — take 1 match in the (=,=) case, and 1 in the (=,) case,

I_2 — take 1 match in the (=,=) case, and 2 in the (=,) case,

I_3 — take 2 matches in the (=, =) case.

Whereas II has six strategies:

II_1 — if (=,) take 2 matches, and if (=,−) take 1 from smaller pile,

II_2 — if (=,) take 2 matches, and if (=,−) take 1 from larger pile,
II_3 — if (=,) take 2 matches, and if (=,−) take 2 from larger pile,
II_4 — if (=,) take 1 match, and if (=,−) take 1 from smaller pile,
II_5 — if (=,) take 1 match, and if (=,−) take 1 from larger pile,
II_6 — if (=,) take 1 match, and if (=,—) take 2 from larger pile.

When we combine the strategies we get the following 3 × 6 payoff matrix:

$$\begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \end{array} \begin{array}{c} \begin{array}{cccccc} II_1 & II_2 & II_3 & II_4 & II_5 & II_6 \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix} \end{array} \qquad (2.$$

As an example, if we play I_1 against II_2 we have the sequence:

$$(=,=) \xrightarrow{I} (=,-) \xrightarrow{II} (-,-) \xrightarrow{I} (-,\ ) \xrightarrow{II} (\ ,\ ), \qquad (2.$$

and so I wins and gets payoff +1.

2.3 MAXIMIN CRITERION

Having obtained the payoff matrix, how do we analyse the game? Which strategy is player I or play II likely to choose, and which should we advise him to choose? The important assumption made is th all game-players should be naturally pessimistic, certainly for zero-sum games. Since your oppone is trying to maximise his payoff, then in a zero sum game this means he is also trying to minimi your own payoff. So for each of I's strategies I looks at the minimum payoff he gets using th strategy. For I_i this is $\min_j e_{ij}$. He might well chose the strategy which has the largest of the minimum payoffs. I is maximising his minimum payoff and we call this the **maximin criterio** Using this strategy he can guarantee he will get a payoff of at least v_L, where

$$v_L = \max_i \min_j e_{ij}. \qquad (2.$$

This is the **lower value** of the game.

II does exactly the same, but since we are recording only I's payoff, which is the negative of II payoff, for each of II's strategy II looks at the maximum I gets under this (which is equivalent to tl minimum that II gets using that strategy). For II_j this will be $\max_i e_{ij}$. II then chooses the strategy th minimises this maximum payoff for I, i.e. he is **minimaxing**. Using this strategy, II can guarantee th I will not get a payoff greater than v_U the **upper value** of the game, where

$$v_U = \min_j \max_i e_{ij}. \qquad (2.$$