# Lectures on Kähler Geometry

ANDREI MOROIANU

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# Contents

# Introduction

These notes grew out of my graduate course at Hamburg University in the autumn of 2003. Their main purpose is to provide a quick and modern introduction to different aspects of Kähler geometry. I had tried to make the original lectures accessible to graduate students in mathematics and theoretical physics having only basic knowledge of calculus in several variables and linear algebra. The present notes should (hopefully) have retained this quality.

The text is organized as follows. The first part is devoted to a review of basic differential geometry. We discuss here topics related to smooth manifolds, tensors, Lie groups, principal bundles, vector bundles, connections, holonomy groups, Riemannian metrics, and Killing vector fields.

The reader familiar with the contents of a first course in differential geometry can pass directly to the second part, which starts with a description of complex manifolds and holomorphic vector bundles. Kähler manifolds are then discussed from the point of view of Riemannian geometry. This part ends with an outline of Hodge and Dolbeault theories, and a simple proof of the famous Kähler identities.

In the third part we study several aspects of compact Kähler manifolds: the Calabi conjecture, Weitzenböck techniques, Calabi–Yau manifolds, and divisors.

The material contained in each chapter is equivalent to a ninety-minute lecture. All chapters end with a series of exercises. Solving them may prove to be at least helpful, if not sufficient, for a reasonable understanding of the theory.

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Paris, September 2006

# Part 1

# Basics of differential geometry

CHAPTER 1

# Smooth manifolds

## 1.1. Introduction

A topological manifold of dimension $n$ is a Hausdorff topological space $M$ which locally "looks like" the space $\mathbb{R}^n$. More precisely, $M$ has an open covering $\mathcal{U}$ such that for every $U \in \mathcal{U}$ there exists a homeomorphism $\phi_U : U \to \tilde{U} \subset \mathbb{R}^n$, called a *local chart*. The collection of all charts is called an *atlas* of $M$. Since every open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ itself, the definition above amounts to saying that every point of $M$ has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Examples. 1. The sphere $S^n \subset \mathbb{R}^{n+1}$ is a topological manifold of dimension $n$, with the atlas consisting of the two stereographic projections $\phi_N : S^n \setminus \{S\} \to \mathbb{R}^n$ and $\phi_S : S^n \setminus \{N\} \to \mathbb{R}^n$ where $N$ and $S$ are the North and South poles of $S^n$.

2. The union $Ox \cup Oy$ of the two coordinate lines in $\mathbb{R}^2$ is not a topological manifold. Indeed, for every neighbourhood $U$ of the origin in $Ox \cup Oy$, the set $U \setminus \{0\}$ has 4 connected components, so $U$ can not be homeomorphic to $\mathbb{R}$.

We now investigate the possibility of defining smooth functions on a given topological manifold $M$. If $f : M \to \mathbb{R}$ is a continuous function, one is tempted to define $f$ to be smooth if for every $x \in M$ there exists $U \in \mathcal{U}$ containing $x$ such that the composition

$$f_U := f \circ \phi_U^{-1} : \tilde{U} \subset \mathbb{R}^n \to \mathbb{R}$$

is smooth. In order for this to make sense, we need to check that the above property does not depend on the choice of $U$. Let $V$ be some other element of the open covering $\mathcal{U}$ containing $x$. If we denote by $\phi_{UV}$ the *coordinate change* function

$$\phi_{UV} := \phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \subset \mathbb{R}^n \to \phi_U(U \cap V) \subset \mathbb{R}^n,$$

then

$$f_V = f_U \circ \phi_{UV}.$$

Our definition is thus coherent provided the coordinate changes are all smooth in the usual sense. This motivates the following:

DEFINITION 1.1. *A* smooth (or differentiable) manifold *of dimension $n$ is a topological manifold $(M,\mathcal{U})$ whose atlas $\{\phi_U\}_{U\in\mathcal{U}}$ satisfies the following compatibility condition: for every intersecting $U, V \in \mathcal{U}$, the map between open sets of $\mathbb{R}^n$*

$$\phi_{UV} := \phi_U \circ \phi_V^{-1}$$

*is a diffeomorphism. If this condition holds, the atlas $\{\phi_U\}_{U\in\mathcal{U}}$ is also called a* smooth structure *on $M$. An atlas is called* oriented *if the determinant of the Jacobian matrix of $\phi_{UV}$ is everywhere positive. An* oriented manifold *is a smooth manifold together with an oriented atlas.*

Unless otherwise stated, all smooth manifolds considered in these notes are assumed to be connected.

We have seen that the existence of a smooth structure on $M$ enables one to define smooth functions on $M$. It is straightforward to extend this definition to functions $f : M \to N$ where $M$ and $N$ are smooth manifolds:

DEFINITION 1.2. *Let $(M, \{\phi_U\}_{U\in\mathcal{U}})$, $(N, \{\psi_V\}_{V\in\mathcal{V}})$ be two smooth manifolds. A continuous map $f : M \to N$ is said to be* smooth *if $\psi_V \circ f \circ \phi_U^{-1}$ is a smooth map in the usual sense for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$. A homeomorphism which is smooth, together with its inverse, is called a* diffeomorphism.

DEFINITION 1.3. *Let $M$ be a smooth manifold. A* local coordinate system *around some $x \in M$ is a diffeomorphism between an open neighbourhood of $x$ and an open set in $\mathbb{R}^n$.*

## 1.2. The tangent space

From now on, unless otherwise stated, by manifold we understand a smooth manifold with a given smooth structure on it. In order to do calculus on manifolds we need to define objects such as vectors, exterior forms, etc. The main tool for that is provided by the *chain rule*. Recall that if $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function, its *differential* at any point $x \in U$ is the linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix in the canonical basis is

$$(df_x)_{ij} := \frac{\partial f_i}{\partial x_j}(x).$$

PROPOSITION 1.4. (Chain rule) *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be two open sets and let $f : U \to \mathbb{R}^m$ and $g : V \to \mathbb{R}^k$ be two smooth maps. Then for every $x \in U \cap f^{-1}(V)$ we have*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x. \tag{1.1}$$

In the particular situation where $m = n = k$, $f : U \to V$ is a diffeomorphism and $g = f^{-1}$, the previous relation reads

$$(d(f^{-1}))_{f(x)} = (df_x)^{-1}. \tag{1.2}$$

Let $x$ be a point of some manifold $(M,\mathcal{U})$ of dimension $n$. We denote by $I_x$ the set of all $U \in \mathcal{U}$ containing $x$. On $I_x \times \mathbb{R}^n$ we define the relation "$\sim_x$" by

$$(U,u) \sim_x (V,v) \qquad \Longleftrightarrow \qquad u = (d\phi_{UV})_{\phi_V(x)}(v).$$

By (1.1) and (1.2), "$\sim_x$" is an equivalence relation. An equivalence class is called a *tangent vector* of $M$ at $x$. By the linearity of the differentials $d\phi_{UV}$ we see that the quotient $I_x \times \mathbb{R}^n / \sim_x$ is an $n$-dimensional vector space. This vector space is called the *tangent vector space* of $M$ at $x$ and is denoted by $T_xM$. The tangent vector at $x$ defined by the pair $(U,u)$ is denoted by $[U,u]_x$. For each $U \in \mathcal{U}$ containing $x$, a tangent vector $X \in T_xM$ has a unique representative $(U,u)$ in $\{U\} \times \mathbb{R}^n$. The vector $u \in \mathbb{R}^n$ is the "concrete" representation in the chart $\phi_U$ of the "abstract" tangent vector $X$.

DEFINITION 1.5. *The union of all tangent spaces $TM := \bigsqcup_{x\in M} T_xM$ is called the* tangent bundle *of $M$. We will see later on that $TM$ has a structure of a vector bundle over $M$ and is, in particular, a smooth manifold of dimension $2n$.*

If $M$ and $N$ are smooth manifolds, $f : M \to N$ is a smooth map and $x \in M$, one can define the differential $df_x : T_xM \to T_{f(x)}N$ in the following way: choose local charts $\phi_U$ and $\psi_V$ around $x$ and $f(x)$ respectively and define

$$df_x([U,u]) := [V, d(\psi_V \circ f \circ \phi_U^{-1})_{\phi_U(x)}(u)]. \tag{1.3}$$

Again, the chain rule shows that the definition of $df_x$ does not depend on the choice of the local charts. It is a straightforward exercise in differentials to check the following extension of the chain rule to manifolds:

PROPOSITION 1.6. *Let $M$, $N$ and $Z$ be smooth manifolds and let $f : M \to N$ and $g : N \to Z$ be two smooth maps. Then for every $x \in M$ we have*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

If $f : M \to N$ is a smooth map, the collection $(df_x)_{x\in M}$ defines a map $df : TM \to TN$, called the differential of $f$, which will sometimes be denoted by $f_*$.

A smooth map $f : M \to N$ is called a *submersion* if its differential $df_x$ is onto for every $x \in M$.

Let $M$ be a smooth manifold of dimension $n$. A topological subspace $S \in M$ of $M$ is called a *submanifold* of dimension $k$ if for every $x \in S$ there exists a neighbourhood $U$ of $x$ in $M$ and a local coordinate system $\phi_U : U \to \tilde{U}$ such that $S \cap U = \phi_U^{-1}(\mathbb{R}^k \cap \tilde{U})$. The restriction to $S$ of all such coordinate systems provides a smooth structure of dimension $k$ on $S$.

THEOREM 1.7. (Submersion theorem) *If $f : M \to N$ is a submersion then $f^{-1}(y)$ is a smooth submanifold of $M$ for every $y \in N$.*

Proof. If $y$ does not belong to the image of $f$, there is nothing to prove. Otherwise, let $x \in f^{-1}(y)$. By taking local charts around $x$ and $y$, we can assume that $M$ and $N$ are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Since $df_x$ is onto, there exists a non-vanishing $m \times m$ minor in the matrix $(\partial f^i/\partial x_j)_{1\le i\le m, 1\le j\le n}$. Without loss of generality we might assume that $(\partial f^i/\partial x_j)_{1\le i,j\le m}$ is non-zero at $x$. Consider the map $F : M \to N \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$, $F(z) = (f(z), z_{m+1}, \ldots, z_n)$. The Jacobian of $F$ at $x$ is clearly non-zero, so by the inverse function theorem there exists some open neighbourhood $U$ of $x$ mapped diffeomorphically by $F$ onto some $\tilde{U} \subset \mathbb{R}^n$. By construction $F(f^{-1}(y) \cap U) = (\{y\} \times \mathbb{R}^{n-m}) \cap \tilde{U}$, so we are done. □

## 1.3. Vector fields

Let $M$ be a smooth manifold. Every map $X : M \to TM$ such that $X(x) \in T_xM$ for all $x \in M$ defines, for every local chart $\phi : U \to \tilde{U} \subset \mathbb{R}^n$, a map $X_\phi : \tilde{U} \to \mathbb{R}^n$ by

$$X_\phi(\phi(x)) := d\phi_x(X_x).$$

If all these maps are smooth, we say that $X$ is a (smooth) *vector field* on $M$. For $x \in M$, $X(x)$ (also denoted by $X_x$) is thus a vector in the tangent space $T_xM$. The set of all vector fields on $M$ is a module over the algebra of smooth functions $\mathcal{C}^\infty(M)$ and is denoted by $\mathcal{X}(M)$.

Example. Let $e_i$ denote the constant vector field on $\mathbb{R}^n$ defined by the $i$th element of the canonical base. If $\phi_U : U \to \tilde{U}$ is a local chart on $M$, we define the local vector field $\partial/\partial x_i$ on $U$ by $\partial/\partial x_i(x) := [U, e_i]_x$, i.e.

$$(d\phi_U)_x\left(\frac{\partial}{\partial x_i}(x)\right) = e_i \qquad \forall\ x \in U.$$

Since for every $x \in U$, $(d\phi_U)_x$ is (tautologically) an isomorphism between $T_xM$ and $\mathbb{R}^n$, $\{\partial/\partial x_i(x)\}_{i=1,\ldots,n}$ is a basis of $T_xM$. We say that $\{\partial/\partial x_i\}$ is a *local frame* on $U$.

This notation is motivated by the following:

Theorem 1.8. *If $\mathfrak{D}(\mathcal{C}^\infty(M))$ denotes the Lie algebra of derivations of the algebra of smooth functions on $M$, there exists a natural isomorphism of $\mathcal{C}^\infty(M)$-modules $\Phi : \mathcal{X}(M) \to \mathfrak{D}(\mathcal{C}^\infty(M))$. In particular, $\mathcal{X}(M)$ has a natural Lie algebra structure.*

Proof. First let $\tilde{U}$ be some open set in $\mathbb{R}^n$. If $\tilde{X} = \sum \tilde{X}^i e_i$ is a smooth vector field on $\tilde{U}$ and $f : \tilde{U} \to \mathbb{R}$ is a smooth function, we define another function $\partial_{\tilde{X}} f$ on $\tilde{U}$ by

$$(\partial_{\tilde{X}} f)(x) := df_x(\tilde{X}_x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)\tilde{X}^i(x).$$

Clearly $\partial_{\tilde{X}} f$ is again smooth, and $\tilde{X}$ defines in this way a derivation of $\mathcal{C}^\infty(U)$:

$$\partial_{\tilde{X}}(fg) = d(fg)(\tilde{X}) = (f\,dg + g\,df)(\tilde{X}) = f\partial_{\tilde{X}}(g) + g\partial_{\tilde{X}} f.$$

If $X$ is now a vector field on a smooth manifold $M$ and $f \in \mathcal{C}^\infty(M)$, we define $(\partial_X f)(x) := df_x(X)$. We will sometimes use the notation $\partial_{X_x} f$ rather than $(\partial_X f)(x)$. Using (1.3) for some chart $\phi_U : U \to \tilde{U}$, one can express the restriction of $\partial_X f$ to $U$ as follows:

$$(\partial_X f)(x) = d(f \circ \phi_U^{-1})_{\phi_U(x)}(\tilde{X}) = (\partial_{\tilde{X}}(f \circ \phi_U^{-1}))(\phi_U(x)), \qquad \forall\, x \in U, \tag{1.4}$$

where $\tilde{X} := d\phi_U(X)$ is the vector field on $\tilde{U}$ corresponding to $X$ in the given chart. The previous argument shows of course that $\partial_X f$ is smooth and that $f \mapsto \partial_X f$ is a derivation.

The map $X \mapsto \partial_X$, clearly defines a morphism $\Phi : \mathcal{X}(M) \to \mathfrak{D}(\mathcal{C}^\infty(M))$ of $\mathcal{C}^\infty(M)$-modules. In order to show that $\Phi$ is an isomorphism, we need to make use of so-called *bump functions*. For $x \in \mathbb{R}^n$ and $0 < \varepsilon < \delta$, a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called a bump function if it is identically 1 on the open ball $B(x, \varepsilon)$ and vanishes identically outside $B(x, \delta)$. The existence of such functions will be proved in an exercise.

Let $X \in \mathrm{Ker}(\Phi)$. This means that

$$\partial_X f = 0 \tag{1.5}$$

for all smooth functions $f$ on $M$. The (relative) difficulty here is to construct enough smooth functions on $M$. Let us fix some $x \in M$ and a chart $\phi_U : U \to \tilde{U}$ such that $U$ contains $x$, and denote by $\tilde{x} := \phi_U(x)$. We choose $0 < \varepsilon < \delta$ such that $\overline{B(\tilde{x}, \delta)} \subset \tilde{U}$ and a bump function $\varphi$ on $\tilde{U}$ relative to the data $(\tilde{x}, \varepsilon, \delta)$. For every function $\psi : \tilde{U} \to \mathbb{R}$, the function

$$f(y) = \begin{cases} (\varphi\psi)(\phi_U(y)), & y \in U, \\ 0, & y \in M \setminus U, \end{cases}$$

is by construction a smooth function on $M$. If $\tilde{X} = d\phi_U(X)$ denotes as before the vector field on $\tilde{U}$ which represents $X$ in the chart $U$, then by (1.4) $\partial_{X_x} f = \partial_{\tilde{X}_{\tilde{x}}}(\varphi\psi)$ and from the properties of $\varphi$ we obtain

$$\partial_{\tilde{X}_{\tilde{x}}}(\varphi\psi) = \psi(\tilde{x})\partial_{\tilde{X}_{\tilde{x}}}\varphi + \varphi(\tilde{x})\partial_{\tilde{X}_{\tilde{x}}}\psi = \psi(\tilde{x})d\varphi_{\tilde{x}}(\tilde{X}) + d\psi_{\tilde{x}}(\tilde{X}) = d\psi_{\tilde{x}}(\tilde{X}).$$

Using (1.5) we get $d\psi_{\tilde{x}}(\tilde{X}) = 0$ for every smooth function $\psi$. Taking $\psi = x_i$ shows that $\tilde{X}_i = 0$ at $\tilde{x}$, so finally $X = 0$ at $x$. Since $x$ was arbitrary, this proves that $X$ vanishes on $M$.

It remains to show that every derivation $D$ on $\mathcal{C}^\infty(M)$ is defined by a smooth vector field on $M$. The proof of this fact will be given in an exercise at the end of this chapter. □

From now on we will often identify a smooth vector field $X$ and the corresponding derivation $\partial_X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ on functions. Notice that every tangent vector at some point $X_x \in T_xM$ defines a linear map $\partial_{X_x} : \mathcal{C}^\infty(M) \to \mathbb{R}$ which satisfies

$$\partial_{X_x}(fg) = g(x)\partial_{X_x}(f) + f(x)\partial_{X_x}(g), \qquad \forall\ f, g \in \mathcal{C}^\infty(M).$$

Definition 1.9. *A* path *on a manifold $M$ is a smooth map $c : \mathbb{R} \to M$. The tangent vector to $c$ at $t$, denoted by $\dot{c}(t)$ is by definition the image of the canonical vector $\partial/\partial t \in T_t\mathbb{R}$ through the differential of $c$ at $t$:*

$$\dot{c}(t) := dc_t\left(\frac{\partial}{\partial t}\right).$$

The formula relating the tangent vector at 0 to a path $c$, $V := \dot{c}(0)$, and the corresponding linear map $\partial_V$ on functions is

$$\partial_V f = df_{c(0)}(V) = df_{c(0)}\left(dc_0\left(\frac{\partial}{\partial t}\right)\right) = d(f\circ c)_0\left(\frac{\partial}{\partial t}\right) = (f\circ c)'(0). \quad (1.6)$$

Definition 1.10. *Let $X$ be a vector field on $M$ and let $x$ be some point of $M$. A* local integral curve *of $X$ through $x$ is a local path $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = x$ and $\dot{c}(t) = X_{c(t)}$ for every $t \in (-\varepsilon, \varepsilon)$.*

Proposition 1.11. *Let $X \in \mathcal{X}(M)$ be a smooth vector field on the manifold $M$.*

*(i) For every $x \in M$ there exists $\varepsilon > 0$ such that for every $\delta \le \varepsilon$, there exists a unique integral curve of $X$ through $x$ defined on $(-\delta, \delta)$.*

*(ii) For every $x \in M$ there exists an open neighbourhood $U_x$ of $x$ and $\varepsilon > 0$ such that the integral curve of $X$ through every $y \in U_x$ is defined for $|t| < \varepsilon$.*

*(iii) For $x \in M$, let $U_x$ and $\varepsilon$ be given by (ii). If $t < \varepsilon$ we define the map $\varphi_t : U_x \to M$ by $\varphi_t(y) := c_y(t)$, where $c_y$ is the integral curve of $X$ through $y$. Then we have*

$$\varphi_t \circ \varphi_s = \varphi_{s+t}, \qquad \forall\ |t|, |s|, |s+t| < \varepsilon, \quad (1.7)$$

*on the open set where the composition makes sense.*

*(iv) For every $t < \varepsilon$, the local map $\varphi_t$ is a local diffeomorphism.*

Proof. Let $\phi : U \to \tilde{U}$ be a local chart such that $x \in U$ and let $\tilde{X} = d\phi(X)$ be the vector field induced by $X$ on $\tilde{U}$. By Proposition 1.6, a local path $c$ is a local integral curve of $X$ if and only if the local path $\tilde{c} := \phi\circ c$ is a local integral curve of $\tilde{X}$. Since the statement of the proposition is local, we can therefore assume that $M$ is an open subset of $\mathbb{R}^n$. Let us express $X$ and the local path $c$ in the canonical frame of $\mathbb{R}^n$ by $X = \sum X_i e_i$ and

$c(t) = \sum c_i(t)e_i$. The fact that $c$ is an integral curve for $X$ through 0 is equivalent to the following system of ODEs:

$$\begin{cases} c(0) = x, \\ c_i'(t) = X_i(c(t)). \end{cases} \tag{1.8}$$

The statements $(i)$ and $(ii)$ now follow directly from the theorem of Cauchy–Lipschitz.

If $y \in U_x$, both curves $c_1(t) := \varphi_{s+t}(y)$ and $c_2(t) := \varphi_t \circ \varphi_s(y)$ are integral curves of $X$ through $\varphi_s(y)$, so by $(i)$ they coincide. This proves $(iii)$.

The theorem of Cauchy–Lipschitz actually says that the solution of the system (1.8) is a smooth function with respect to both $x$ and $t$. This shows that each $\varphi_t$ is smooth. Finally, $(iv)$ follows by taking $t = -s$ in $(iii)$ and using that $\varphi_0$ is the identity map by definition. □

A family of local diffeomorphisms $\{\varphi_t\}$ of $M$ satisfying (1.7) is called a *pseudogroup of local diffeomorphisms* of $M$. If the local maps $\varphi_t$ are defined by a vector field $X$ as before, the pseudogroup $\{\varphi_t\}$ defined in Proposition 1.11 is called the *local flow* of $X$.

A vector field is called *complete* if its flow is globally defined on $M \times \mathbb{R}$.

We will use the flow of vector fields in order to differentiate interesting objects on smooth manifolds called tensor fields. In order to introduce them, we need to recall some background of linear algebra in the next chapter.

## 1.4. Exercises

(1) Show that a connected topological manifold is path connected.

(2) Use the explicit formulas of the stereographic projections to check that the sphere $S^n$ is a smooth manifold.

(3) Alternatively, show that $S^n$ is a smooth manifold using the submersion theorem.

(4) Show that every connected component of a smooth manifold is again a smooth manifold.

(5) (*The real projective space*) Let $\mathbb{R}\mathrm{P}^n$ denote the space of real lines in $\mathbb{R}^{n+1}$ passing through 0. Check that $\mathbb{R}\mathrm{P}^n$ is a smooth manifold of dimension $n$. *Hint:* Denote by $U_i$ the set of lines not contained in the hyperplane $(x_i = 0)$. Show that there exist natural bijections $\phi_i : U_i \to \mathbb{R}^n$ defining a topology and a smooth atlas on $\mathbb{R}\mathrm{P}^n$.

(6) If $M$ and $N$ are smooth manifolds, show that $M \times N$ has a smooth structure such that the canonical projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are smooth.

(7) Let $M$ and $N$ be smooth manifolds and let $x \in M$, $y \in N$ be arbitrary points. Show that the tangent space $T_{(x,y)}M \times N$ is naturally isomorphic to $T_xM \oplus T_yN$.

(8) Prove that if $\phi_U : U \to \tilde{U}$ is a local chart on a manifold $M$ and $X = [U, u]$ is an abstract tangent vector at some $x \in U$, then $(d\phi_U)_x(X) = u$. *Hint:* Since $\tilde{U}$ is an open subset of $\mathbb{R}^n$, one may apply (1.3) by choosing the trivial chart $\psi_V = \mathrm{Id}_{\tilde{U}}$.

(9) Let $\{\partial/\partial x_i\}$ and $\{\partial/\partial y_i\}$ be the local frames on $M$ defined by two local charts $\phi : U \to \tilde{U}$ and $\psi : V \to \tilde{V}$. We denote by $\{x_i\}$ and $\{y_i\}$ the coordinates on $\tilde{U}$ and $\tilde{V}$ respectively, and by a slight abuse of notation, we denote by $x = x(y)$ the diffeomorphism $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$. Prove the relation

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$$

between local vector fields on $M$.

(10) Let $\varphi_t$ be the local flow of a vector field $\xi$. Prove that $(\varphi_t)_*(\xi_x) = \xi_{\varphi_t(x)}$ for all $t$ and $x$ where $\varphi_t(x)$ is defined.

(11) Let $f : \mathbb{R} \to \mathbb{R}$ be the smooth function

$$f(t) := \begin{cases} e^{-\frac{1}{t(1-t)}}, & t \in (0,1), \\ 0, & \text{otherwise}, \end{cases}$$

and denote by $F$ the "normalized" primitive of $f$:

$$F(s) := \frac{\int_s^\infty f(t)dt}{\int_{\mathbb{R}} f(t)dt}.$$

Prove that for $0 < \varepsilon < \delta$, the function $\varphi : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\varphi(x) := F\left(\frac{|x|^2 - \varepsilon^2}{\delta^2 - \varepsilon^2}\right)$$

is a bump function which is identically 1 on the open ball $B(0, \varepsilon)$ and vanishes identically outside $B(0, \delta)$.

(12) Show that for every derivation $D$ of $\mathcal{C}^\infty(M)$ there exists a smooth vector field $X$ such that $D = \partial_X$. *Hint:* Let $p \in M$ and let $(x_1, \dots, x_n) = \phi_U : U \to \tilde{U}$ be a local coordinate system around $p$. If $\varphi$ is a bump function around $p$ with support contained in $U$, define

$$X_p := \sum_{i=1}^{n} D(\varphi x_i) \left( \frac{\partial}{\partial x_i} \right)_p$$

and show that $X_p$ is independent of $\varphi$ and $\phi_U$. Check the smoothness of the vector field obtained in this way.

(13) Show that every vector field on a compact manifold is complete.