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Topological Methods in the Study of Boundary Value Problems

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Preface

This book constitutes an elementary introduction to the application of topological techniques in nonlinear analysis. For the reader's convenience, only boundary value problems for ordinary differential equations are treated, although most of the ideas can be generalized for partial differential equations and some other areas of mathematics. This approach will allow the student to avoid many of the technical difficulties related to nonlinear problems and focus on the application of topological methods. Only basic knowledge of the main topics in analysis is needed, making the book easy to understand for nonspecialists, too. Whenever possible, just elementary tools are employed; in particular, in many situations we arrive at important and nontrivial results by means of elementary techniques. Despite its simplicity, the main ingredients of nonlinear analysis are present, so readers with some experience in functional analysis or differential equations may also find some elements that complement and enrich their tools for solving nonlinear problems in many different fields. The style throughout the book is concise and informal, allowing students of all levels to have a first glimpse at this interesting and beautiful branch of mathematics.

This book could never have existed without the support of my colleagues, students, and friends (most of them belong to at least two of the listed categories): Rafael Ortega, Mónica Clapp, Colin Rogers, Man Kam Kwong, Alfonso Castro, Lev Idels, Jorge Cossio, Julián Haddad, Pablo De Nápoli, Juan Pablo Pinasco, Diego Rial, Paula Kuna, Alberto Déboli, Manuel Maurette, and Rocío Balderrama. All that I've learned about mathematics is due to them. Also, I'm grateful for the support of my family and the team of "nonmathematical" friends from whom I've learned all that I know about life. I want to give special thanks to Donna Chernyk for trusting in this project and for all her help and patience and to the anonymous reviewers for all the corrections and comments that helped to improve the first version of this manuscript. All remaining mistakes and flaws of the text are my sole responsibility.

Buenos Aires, Argentina — Pablo Amster

Introduction

Nonlinear analysis is a field with a large number of applications in various sciences. In particular, the study of boundary value problems for differential equations has been the subject of intense research in recent decades. Many different techniques have been developed for the study of nonlinear problems; among them, one of the most effective techniques consists in the use of diverse topological methods, such as the shooting method, fixed point theorems, upper and lower solutions, or degree theory. In particular, topological methods have proved to be successful for many problems that have no variational structure. A first basic result in the direction of the fixed point methods is the now well-known Banach fixed point theorem, which generalizes the method of successive approximations proposed by Picard. Despite its simplicity (and the fact that it is almost 100 years old), the Banach theorem is still an efficient and popular tool for proving many different existence-uniqueness results. However, in some cases the conditions of this theorem are too restrictive, and more powerful techniques are required. An example is the Schauder fixed point theorem, which can be regarded as an extension, for a compact operator in a Banach space, of the well-known Brouwer theorem and is especially useful in the field of differential equations, where the associated operators usually have a compact inverse in some appropriate space. The Schauder theorem can also be used to develop the method of upper and lower solutions, which makes it possible to prove the existence and location of solutions for a very general class of equations. Also, there are some fixed point theorems in cones of the Krasnoselskii type that have applications to some specific problems, for instance, many proofs of existence of positive solutions of some differential equations, frequently found in real-world applications, are based on these kinds of results.

All the aforementioned techniques can be introduced within the more general scope of topological degree theory, which, roughly speaking, is an algebraic count of the zeros of a continuous function defined over a bounded subset of a normed space. In the finite-dimensional case, it was defined by Brouwer and provides a straightforward proof of his fixed point theorem, among other results. The extension to a general Banach space is due to Leray and Schauder and assumes that the function is a compact perturbation of the identity, namely, an operator of the type $I-K$, with K compact. Again, this setting can be regarded as "natural" in the context of many boundary value problems for differential equations. Besides the solution property, which ensures that a mapping with nonzero degree has a zero, one of the most powerful properties of the degree is its homotopy invariance that makes it possible, under appropriate conditions, to transform a problem into a simpler one for which the degree is easy to compute. The equation under study is thus embedded into a one-parameter family of equations; the existence of a priori bounds of the solutions guarantees that the degree will be constant over the deformation. The topological degree is particularly useful in so-called *resonant problems*, those in which the associated linear operator is noninvertible, and hence it is not always clear how to convert them into a fixed point equation.

The book is self-contained, in the sense that only basic notions of analysis are needed to understand most of the contents. The examples mainly concern boundary value problems for ordinary differential equations. In most cases, we shall take as a model equation the second-order problem

$$u''(t) = f(t,u(t),u'(t)), \qquad 0 < t < 1,$$

with f continuous, under the boundary conditions

$$u(0) = u(1) = 0 \qquad \text{(Dirichlet)},$$

$$u'(0) = u'(1) = 0 \qquad \text{(Neumann)},$$

or

$$u(0) = u(1), \qquad u'(0) = u'(1) \qquad \text{(periodic)},$$

among others. The latter conditions can be interpreted as truly periodic when $f(t+1,u,v) = f(t,u,v)$ for all u,v, and $t \in \mathbb{R}$: in this case, a solution $u : [0,1] \to \mathbb{R}$ can be extended periodically to a C^2 function defined in the whole real line. This setting will be particularly useful when dealing with some *delay differential equations*, in which the nonlinear term f also depends on $u(t-\tau)$ for some $\tau > 0$. In some cases, our model equation will in fact be a *system* of n equations: as we shall see, this extension is not always trivial and may involve some geometrical or topological difficulties.

The results presented here are not the best of all possible results, in the sense that, in most cases, we prioritized giving an intuitive and easy approach over obtaining better or sharper theorems. We focus all the time on the methods and on the specific problems; in particular, this is one of the reasons for which all examples refer only to ordinary differential equations, which makes it possible to avoid some technicalities. Also, we have chosen to work always in the spaces of continuous or continuously differentiable functions, although better results can be obtained using, for example, Sobolev spaces. In many cases, the same problem is studied using different tools, so it may happen sometimes that a result presented in one chapter is improved in a later one. The reader may also feel that some of the computations required for the different methods are unnecessarily repeated in different chapters, but this was done for the sole purpose of preserving the "self-containedness" spirit of the text.

The book is organized as follows. In Chap. 1, we introduce one of the simplest topological methods, usually known as the *shooting method*, which basically consists in reducing a problem to a finite-dimensional equation for a certain parameter λ. Then, appropriate tools can be used, such as the Brouwer fixed point theorem or equivalent results. The chapter was designed to be self-contained and employs only concepts from basic calculus; for simplicity, our study of systems is restricted here to the two-dimensional case, for which we present a very elementary proof of the fixed point theorems we shall be using. There are many extensions and improvements of the basic results, which require slightly more advanced topics (e.g., Stone–Weierstrass theorems); for this reason, they were included in starred sections. This does not mean, of course, they are extremely difficult: the idea was just to show that most of the topics—the nonstarred sections—could be understood within the context of a first course in calculus.

The next chapter is devoted to the Banach fixed point theorem and some of its immediate consequences. In particular, we shall prove the usual version of the implicit function theorem in Banach spaces and present some applications to boundary value problems. This requires a knowledge of the basic notions of differentiation in Banach spaces, which for the sake of completeness are presented here.

In Chap. 3 we develop some iterative methods such as the monotone iterations method and the Newton method and some of its variants. Applications are given to some boundary value problems. Also, we introduce a Cantor diagonalization argument, which makes it possible to deal with problems in unbounded domains.

In Chap. 4 we prove the general version of the Brouwer theorem and a well-known extension to Banach spaces: the Schauder theorem. Among other uses, this latter result allows us to give a complete version of the method of upper and lower solutions introduced in the previous chapter, with applications to many different problems. As a corollary, we obtain the so-called Schaefer theorem, which can be regarded as a particular case of the Leray–Schauder continuation techniques.

These techniques require a more sophisticated topological tool: the aforementioned topological degree, constructed with some detail in Chap. 5. As usual, the Brouwer degree is introduced first and then extended for compact perturbations of the identity in a Banach space, namely, the Leray–Schauder degree. The specific difficulties of the construction are not essential for the applications, so readers not particularly interested in certain topological issues may avoid most of the contents in this chapter and keep in mind only the main properties of the degree mapping.

Finally, in Chap. 6 we present applications to various boundary value problems. Starting with specific examples, we obtain some general continuation theorems that can be applied in many situations. In particular, most of the sections are devoted to the study of resonant problems, for which we discuss some classical results and different extensions.

For the reader's convenience, we include a brief review of the main results from the theory of ordinary differential equations used in the book. The list reduces to some fundamental theorems such as existence and uniqueness, continuous dependence, and a few specific facts concerning second-order equations. Also, we give an account of some definitions and elementary aspects of the general theory of metric spaces. Some specific results used in various chapters are commented upon in the respective appendices.

Incidentally, a few basic facts of measure theory shall be mentioned or employed in some particular examples, such as the Lebesgue measure of a set or some well-known convergence theorems like the dominated convergence theorem or the Fatou lemma. In a (unique) specific exercise we refer to convolution and mollifiers, but all of this can be completely skipped by the uninterested reader.

All chapters end with a list of exercises. For convenience, there is a section at the end of the book with hints and solutions to selected problems. Every exercise, no matter how easy, deserved at least one or two lines in this latter section: sometimes just a short comment might be of some help or provide a different point of view. Complete resolution is not the general rule, although some more difficult problems are solved in a reasonably detailed manner.

To conclude this introduction, it is worth mentioning that the bibliography at the end of the book includes not only the various papers and books cited in the different chapters but also a rapid account of general texts that cover most of the topics addressed in the text e.g. [3, 4, 22, 26, 32, 81, 93, 104, 107]. The list is not exhaustive; it was intended to provide a basic overview of the various paths the reader might take to tackle more in-depth studies on the subject.

Contents

Notation and Special Symbols

Throughout the book, we shall use the standard terminology for sets and relations. In particular, we shall use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ for natural, whole, real, and complex numbers, respectively. Also, we shall use $'$ for derivatives of a real function and $\frac{\partial}{\partial x_j}$ for partial derivatives. The notation "D" is reserved for the differential of a mapping. Other typical symbols and notations shall appear, such as, for example, lim, lim sup, and lim inf. The closure of a subset A of a metric space X shall be denoted by $\overline{A}$. The distance between two elements $x, y \in X$ shall be denoted by $d(x,y)$. An open ball of radius r centered at a point $x \in X$ shall be denoted by $B_r(x)$. In a few specific situations we shall employ the distance between a point and a subset $A \subset X$, denoted by $dist(x,A)$. If L is a linear operator between two vector spaces, we shall use the standard notation $Ker(L)$ and $Im(L)$ to refer to the kernel and range or image of L, respectively. In the case of a matrix $A \in \mathbb{R}^{m\times n}$, we shall write a_{ij} for its ij entry. The isomorphism between $\mathbb{R}^{m\times n}$ and the space $L(\mathbb{R}^n, \mathbb{R}^m)$ of linear transforms from $\mathbb{R}^n$ to $\mathbb{R}^m$ shall be ignored: in other words, for $x \in \mathbb{R}^n$, we shall avoid the use of transpose symbols and write directly $Ax \in \mathbb{R}^m$. When $n = m$, the determinant of A shall be denoted by $det A$.

Unless specified, when X is a normed space, the norm of an element $x \in X$ shall be written, as usual, $\|x\|$. Special cases include the following:

- $\mathbb{R}^n$, for which we shall use the notation $|\cdot|$;
- The space $C([a,b])$ of continuous vector functions $u : [a,b] \to \mathbb{R}^n$, with norm

$$\|u\|_\infty := \max_{t\in[a,b]} |u(t)|.$$

  More generally, for $k > 0$ we shall denote by $C^k([a,b])$ the Banach space of C^k functions from $[a,b]$ to $\mathbb{R}^n$, equipped with the norm

$$\|u\|_{C^k} := \max\{\|u\|_\infty, \|u'\|_\infty, \ldots, \|u^{(k)}\|_\infty\}.$$

  Occasionally, we shall employ the space C_T of continuous T-periodic functions $u : \mathbb{R} \to \mathbb{R}^n$, also equipped with a uniform norm;

- The usual Lebesgue space $L^2(a,b)$ of measurable vector functions $f:(a,b)\to\mathbb{R}^n$ such that

$$\|f\|_{L^2} := \left(\int_a^b |f|^2\right)^{1/2} < \infty.$$

In general, an inner product defined over a vector space shall be denoted by $\langle\cdot,\cdot\rangle$. This includes the case of $L^2(a,b)$, with

$$\langle f,g\rangle := \int_a^b f\cdot g.$$

For the euclidean inner product of two elements $x,y\in\mathbb{R}^n$ we shall directly use the dot notation $x\cdot y$, that is,

$$x\cdot y := \sum_{j=1}^n x_j y_j.$$

The maximum, minimum, and average of a continuous function $u:[a,b]\to\mathbb{R}$ shall be denoted respectively by u_{max}, u_{min}, and $\overline{u}$, namely,

$$u_{max} := \max_{t\in[a,b]} u(t), \qquad u_{min} := \min_{t\in[a,b]} u(t), \qquad \overline{u} := \frac{1}{b-a}\int_a^b u(t)\,dt.$$

When $u:[a,b]\to\mathbb{R}^n$, we shall use the same notation $\overline{u}$ to denote its average coordinate by coordinate, that is,

$$\overline{u} := (\overline{u_1},\ldots,\overline{u_n}).$$

Some specific terminology and symbols such as *deg* for the degree of a function, *co* for the convex hull of a set, or *meas* for measure shall be defined within the particular context in which it is used.

Chapter 1
Shooting Type Methods

1.1 Set the Angle and Shoot

In this chapter, we shall describe a very elementary tool for the study of boundary value problems, usually known as the *shooting method*. In very general terms, the method can be summarized in two steps:

1. Solve an initial value problem with a free parameter λ.
2. Find an appropriate value of λ such that the obtained solution satisfies the desired boundary condition.

The task looks really simple, but it requires some qualitative analysis on the behavior of the solutions of the initial value problem, according to the variations in the parameter λ. As we shall see, in many cases it is possible to obtain enough information in advance to guarantee the success of the method.

Let us start with the second-order equation

$$u''(t) = f(t, u(t)) \tag{1.1}$$

with homogeneous Dirichlet conditions

$$u(0) = u(1) = 0. \tag{1.2}$$

Assume we know little about boundary value problems at this point, so we should start doing what we undoubtedly know: solve, in the first place, on Eq. (1.1) with initial conditions

$$u(0) = 0, \qquad u'(0) = \lambda \tag{1.3}$$

for fixed $\lambda \in \mathbb{R}$. Then, hopefully, we will be able to find a value of the shooting parameter λ such that the corresponding solution will also satisfy the boundary condition at $t = 1$.

To do things properly, assume that f is continuous and locally Lipschitz in u; then the solution u_λ of (1.1)–(1.3) is well defined and unique. Thus, it suffices to find λ such that $u_\lambda(1) = 0$; in other words, we are looking for a zero of the function ϕ defined by

$$\phi(\lambda) := u_\lambda(1).$$

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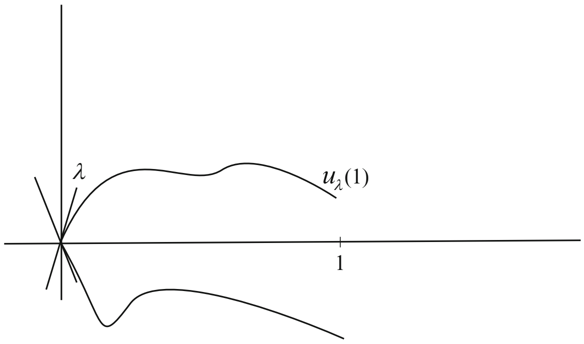

This explains the title of this section: indeed, the procedure simply represents an attempt to adjust the value of the parameter λ until an appropriate shooting angle is obtained, so the graph of u_λ hits the point $(1,0)$. This recalls somewhat an old computer game called *Gorilla*, now replaced by the very popular (and slightly more sophisticated) *Angry Birds*.

But there is always a *but*: observe that the solutions of the initial value problem may not be defined up to $t = 1$, and hence ϕ is not necessarily defined for all values of λ. However, we do know at least one very important fact about ϕ: it is continuous over its domain, no matter what this domain looks like.

In some lucky situations, it is possible to assert that $dom(\phi) = \mathbb{R}$; for example, this is the case when f grows at most linearly in its second variable, that is,

$$|f(t,u)| \leq a|u| + b$$

for some constants a and b (see Appendix B.1, Exercise 4 for details). In particular, if f is bounded, then the shooting method works perfectly well, as we shall see in the next section.

1.1.1 A Back-and-Forth Example: The Pendulum Equation

Let us illustrate the technique described previously with a well-known problem, the pendulum equation

$$u''(t) + \sin u(t) = p(t), \qquad 0 < t < 1,$$

where the forcing term $p : [0,1] \to \mathbb{R}$ is continuous. This equation is very famous; it has been the subject of many relevant works and is still being studied: for instance, the problem of finding all possible forcing terms such that periodic solutions exist has not yet been solved completely (for an account of the history and open problems on the pendulum equation, see, for example, the excellent survey [79]. More shall be said about the subject later on, in Chap. 3). However, the situation is completely different when dealing with the Dirichlet conditions (1.2), for which the existence of solutions can be easily proven by the shooting method.

Indeed, if u_λ is the unique solution satisfying the initial value conditions (1.3), then integration yields

$$u'_\lambda(t) = \lambda + \int_0^t [p(s) - \sin u_\lambda(s)]\,ds$$

and, hence,

$$|u'_\lambda(t) - \lambda| \leq \int_0^t |p(s) - \sin u_\lambda(s)|\,ds \leq \int_0^1 |p(t)|\,dt + 1.$$

Setting $R := \int_0^1 |p(t)|\,dt + 1$ we deduce that u_λ is monotone for $|\lambda| \geq R$, more precisely:

- $\lambda \geq R \Rightarrow u'_\lambda(t) \geq 0$ for all t,
- $\lambda \leq -R \Rightarrow u'_\lambda(t) \leq 0$ for all t.

In particular, u_R is nondecreasing and u_{-R} is nonincreasing; together with the fact that $u_\lambda(0) = 0$ and $u'_\lambda(0) = \lambda$, this implies

$$\phi(R) > 0 > \phi(-R).$$

Thus, by Bolzano's theorem we conclude that ϕ vanishes in $(-R, R)$.

The same procedure can be applied to Eq. (1.1) for arbitrary bounded f. But the boundedness condition is too restrictive: if we are willing to say that the shooting method is a useful tool for many different problems, then it would be desirable to see it applied in more general cases. This is the goal of the next section.

1.1.2 A Priori Bounds

From the previous example we may conclude, as a general rule, that the success of the shooting method relies very strongly on the fact that we know some properties of the associated flow of the differential equation. For instance, we need to be sure that the solutions of the initial value problem for an appropriate set of parameters are defined up to the endpoint of the interval. As mentioned, from standard results in the theory of ordinary differential equations, this is guaranteed for all λ when the nonlinearity has linear growth; however, there are plenty of cases in which this restriction is not satisfied and the shooting method is still applicable. In particular, in some situations it might be very helpful to be equipped with count with a priori bounds of the solutions.

The philosophy behind this idea is again very simple: if we know in advance that the solutions of a certain problem are bounded by some constant R, then we may replace the nonlinearity f by a bounded one, say $\tilde{f}$, such that $\tilde{f}(t,u) = f(t,u)$ for $|u| \leq R$. Obviously, this must be done in such a way that the solutions of the modified problem $u'' = \tilde{f}(t,u)$ are also bounded by the same R, so we can ensure that they are in fact solutions of the original problem. Let us consider some basic examples.

Example 1.1. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ of class be continuous and locally Lipschitz in its second variable and assume there exists a positive constant $R > 0$ such that

$$f(t,-R) < 0 < f(t,R) \qquad \text{for all } t \in [0,1]. \tag{1.4}$$

Then the Dirichlet problem (1.1)–(1.2) has at least one solution u with $\|u\|_\infty \leq R$.

Condition (1.4) is a particular case of the so-called *Hartman condition*, which shall be introduced subsequently. Here, it seems reasonable to define

$$\tilde{f}(t,u) := \begin{cases} f(t,u) & \text{if } |u| \leq R, \\ f(t,R) & \text{if } u > R, \\ f(t,-R) & \text{if } u < -R. \end{cases}$$

This "cutoff" operation might look a bit drastic; nevertheless, it is true that $\tilde{f}$ is still continuous and locally Lipschitz in u. Moreover, it is bounded, so the shooting method provides a solution u of the problem $u''(t) = \tilde{f}(t,u)$ satisfying (1.2). Thus, it suffices to prove that $|u(t)| \leq R$ for all t. Suppose, for example, that u achieves its maximum value at some t_0 with $u(t_0) > R$; then $t_0 \in (0,1)$ and

$$u''(t_0) = \tilde{f}(t_0, u(t_0)) = f(t_0, R) > 0,$$

a contradiction. The proof that $u(t) \geq -R$ for all t is analogous. Note that, in this case, we have proven that the norm of an arbitrary solution of the problem with $\tilde{f}$ is bounded by R, although the original problem might also have other solutions.

Example 1.2. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and locally Lipschitz in its second variable, and assume that f is nondecreasing with respect to u, that is,

$$f(t,u) \leq f(t,v) \quad \text{for all } t \in [0,1], u,v \in \mathbb{R},\ u \leq v.$$

Then the Dirichlet problem (1.1)–(1.2) has a unique solution.

This problem looks more tricky, but it can still be solved using only elementary arguments. In the first place, we should observe a remarkable novelty with respect to the preceding examples: here, the solution is unique. Thus it seems like a good idea to try to understand the role of the monotonicity condition in order to see how uniqueness follows from it.

To this end, note that if u and v are solutions of the problem, then the function $w := u - v$ satisfies

$$w''(t)w(t) = [f(t,u(t)) - f(t,v(t))][u(t) - v(t)] \geq 0.$$

Moreover, $w(0) = w(1) = 0$, so integration of the previous inequality yields

$$\int_0^1 w'(t)^2\,dt = -\int_0^1 w''(t)w(t),\ dt \le 0.$$

This implies that $w' \equiv 0$, which in turn implies $w \equiv 0$.

At this point, we might say: "Great, we have proven uniqueness, but... how do we now deduce the *existence* of solutions?" As we shall see in subsequent chapters, this is a particular case of a rather general class of problems in which, roughly speaking, *uniqueness implies existence*. As before, we shall replace f by a cutoff function $\tilde{f}$; however, in the present case we do not have, for the moment, a value R on which to cut. Thus, we must choose it in an accurate way.

That is the key idea of the aforementioned a priori bounds: sometimes it is possible, before knowing whether or not the problem has a solution, to prove that the norm of such a solution, if it does exist, is smaller than a certain constant. Here, if u solves (1.1)–(1.2), then we may write

$$u''(t) = f(t,u(t)) - f(t,0) + f(t,0)$$

and, since $u(0) = 0$, we obtain

$$u''(t)u(t) = [f(t,u(t)) - f(t,0)]u(t) + f(t,0)u(t) \ge f(t,0)u(t).$$

Thus, integration at both sides of the inequality yields

$$\int_0^1 u'(t)^2\,dt \le -\int_0^1 f(t,0)u(t)\,dt \le \|u\|_\infty \int_0^1 |f(t,0)|\,dt. \tag{1.5}$$

Next, recall the standard notation

$$A^+ = \max\{A,0\}, \qquad A^- = \max\{-A,0\},$$

and write $u(t) = \int_0^t u'(s)\,ds$ to deduce that

$$-\int_0^1 [u'(s)]^-\,ds \le -\int_0^t [u'(s)]^-\,ds \le u(t) \le \int_0^t [u'(s)]^+\,ds \le \int_0^1 [u'(s)]^+\,ds$$

for all t. Furthermore, since $u(0) = u(1)$,

$$0 = \int_0^1 u'(s)\,ds = \int_0^1 ([u'(s)]^+ - [u'(s)]^-)\,ds,$$

so

$$\int_0^1 [u'(s)]^-\,ds = \int_0^1 [u'(s)]^+\,ds = \frac{1}{2}\int_0^1 ([u'(s)]^+ + [u'(s)]^-)\,ds = \frac{1}{2}\int_0^1 |u'(s)|\,ds.$$

From the previous inequalities, $|u(t)| \le \frac{1}{2}\int_0^1 |u'(s)|\,ds$, and hence, by the Cauchy–Schwarz inequality, we conclude that

$$\|u\|_\infty \leq \frac{1}{2}\|u'\|_{L^2}.$$

Combined with (1.5), this implies

$$\|u\|_\infty \leq \frac{1}{2}\|u'\|_{L^2} \leq \frac{1}{2}\int_0^1 |f(t,0)|\,dt := R.$$

If we set $\tilde{f}$ exactly as in the previous example, then the modified problem has a solution u. But now comes the most interesting part: the bound R was obtained *using only the monotonicity of* f. Thus, because $\tilde{f}$ is also nondecreasing in its second variable and $\tilde{f}(t,0) = f(t,0)$, we deduce that $\|u\|_\infty$ is bounded by the same R, and hence u is a solution of the original problem.

Example 1.3. Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous and locally Lipschitz in its second variable, and assume that f has *linear growth* in u, that is, $|f(t,u)| \leq A|u| + B$ for some constants $A, B \geq 0$. We claim that, if $A < 4$, then the the Dirichlet problem (1.1)–(1.2) has a solution. Indeed, in this case, an a priori bound for an arbitrary solution u is obtained as follows. As before, let us multiply by $u(t)$ and integrate to obtain

$$\int_0^1 u'(t)^2\,dt = -\int_0^1 f(t,u(t))u(t)\,dt \leq A\int_0^1 u(t)^2\,dt + B\int_0^1 |u(t)|\,dt.$$

The computations in the previous example now yield

$$\|u\|_\infty^2 \leq \frac{1}{4}\|u'\|_{L^2}^2 \leq \frac{A\|u\|_\infty^2 + B\|u\|_\infty}{4},$$

and hence $\|u\|_\infty \leq \frac{B}{4-A}$. Fix $R \geq \frac{B}{4-A}$, and let $\tilde{f}$ be defined as previously. Moreover, $|\tilde{f}(t,u)| \leq A|u| + B$, so the solutions of the truncated problem are, in particular, solutions of the original one.

Remark 1.1. The condition $A < 4$ can be improved; indeed, using the last inequality in Appendix B, Sect. B.2.2, it is verified that

$$\|u\|_\infty \leq \frac{1}{8}\|u''\|_\infty \leq \frac{A\|u\|_\infty + B}{8},$$

and it suffices to assume $A < 8$. Furthermore, from the methods that shall be introduced in subsequent chapters, it is possible to prove the existence of solutions under the even weaker condition $A < \pi^2$. This latter assumption is already sharp, as may be seen from the following example. Let $f(t,u) = \sin(\pi t) - \pi^2 u$, and suppose that (1.1)–(1.2) has a solution u. Multiply by $\sin(\pi t)$ at both sides and integrate by parts to obtain

$$\int_0^1 \sin^2(\pi t)\,dt = \int_0^1 [u''(t) + \pi^2 u(t)]\sin(\pi t)\,dt = 0,$$

a contradiction.

1.2 From Scalar Equations to Systems

Besides the extensions obtained in previous examples, the application of the shooting method to the case of a bounded nonlinearity might have been a bit disappointing for those expecting something really spectacular. In some sense, the resolution was just *too* simple: notice, for example, that our conclusions have relied only on the fact that if the absolute value of the first derivative of a solution u at the initial point $t = 0$ is large, then u is monotone. This is, indeed, very simple, although it is not clear how it might be extended for a *system* of equations.

The goal of this section consists in solving, instead of a scalar equation, a system of two equations. In other words, we shall still consider Eq. (1.1), but now with $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$, and look for a solution $u : [0,1] \to \mathbb{R}^2$ satisfying the Dirichlet conditions (1.2).

Of course, it is natural to ask why we should restrict ourselves to the two-dimensional case. In fact, all the results in this section can be extended to higher dimensions; however, systems of two equations already contain the main ingredients of the nonscalar case and have the additional advantage of allowing a very elementary and elegant approach. Analogous conclusions for the general case can be easily obtained from the results that shall be developed later, in Chap. 4.

1.2.1 With a Little Help from My Intuition

With the scalar case still in mind, let us now assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and locally Lipschitz in u and define, for $\lambda \in \mathbb{R}^2$, the function u_λ as the unique solution of the initial value problem (1.1)–(1.3). As before,

$$u'_\lambda(t) = \lambda + \int_0^t f(s, u(s))\, ds.$$

But this time we may perform a second integration step to obtain

$$u_\lambda(t) = \lambda + \int_0^t \int_0^\tau f(s, u_\lambda(s))\, ds\, d\tau.$$

As long as u_λ is still defined for $t = 1$, we may set again $\phi(\lambda) := u_\lambda(1)$, and hence

$$\phi(\lambda) = \lambda + S,$$

where S (for “something”) satisfies $|S| \leq \|f\|_\infty$.

The bad news is that, in this new context, the Bolzano theorem cannot be used since we are now dealing with a two-dimensional problem. Consequently, the existence of a zero of ϕ is not yet obvious.

Let us first try a simple argument: in the first place, it is clear that if $R > \|f\|_\infty$, then $|\phi(\lambda) - \lambda| < R$ for all λ. Thus,